

Spherical Lagrangians via ball packings and symplectic cutting

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Abstract

In this paper we prove the connectedness of symplectic ball packings in the complement of a spherical Lagrangian, S^2 or \mathbb{RP}^2 , in symplectic manifolds that are rational or ruled. Via a symplectic cutting construction this is a natural extension of McDuff's connectedness of ball packings in other settings and this result has applications to several different questions: smooth knotting and unknottedness results for spherical Lagrangians, the transitivity of the action of the symplectic Torelli group, classifying Lagrangian isotopy classes in the presence of knotting, and detecting Floer-theoretically essential Lagrangian tori in the del Pezzo surfaces.

1 Introduction

In [28] the second and third named authors investigated the existence and uniqueness (unknottedness) problems of Lagrangian S^2 in rational manifolds via tools from symplectic field theory and the study of symplectic ball-packings [37, 34]. In this paper we continue to explore the connections between symplectic ball packing, symplectic cutting, and Lagrangian unknottedness, while answering several questions from [28] and extending the results to Lagrangian \mathbb{RP}^2 s. See [9] for an early survey of the problem of Lagrangian knots and more recent results in [5, 8, 11, 12, 16, 40, 41].

Our first result, conjectured in [28, Remark 5.2], is on the connectedness of symplectic ball packings in the complement of a Lagrangian S^2 or \mathbb{RP}^2 :

Theorem 1.1. *Let (M^4, ω) be a closed 4-dimensional symplectic manifold that is rational or ruled and let $L \subset M$ be a Lagrangian S^2 or \mathbb{RP}^2 , then the space of symplectic ball packings in $M \setminus L$ is connected.*

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Via symplectic cutting, Theorem 1.1 follows from the connectedness of the space of ball packings in the complement of a symplectic sphere. This is established for certain symplectic surfaces in Proposition 2.1 by work of McDuff on relative inflation [34, 36].

Recall that a symplectic rational manifold (M^4, ω) is where M is either \mathbb{CP}^2 , a symplectic blow-up of \mathbb{CP}^2 , or $S^2 \times S^2$. In [28], building on work of Evans [11], Hamiltonian unknottedness for Lagrangian S^2 's in symplectic rational manifolds was established when the Euler characteristic $\chi(M) \leq 7$, except for the case of a characteristic homology class. We complete the picture here in Theorem A.2 in the appendix. This result is sharp due to Seidel's [40] construction of Hamiltonian knotted Lagrangian spheres in $\mathbb{CP}^2 \# 5\overline{\mathbb{CP}}^2$.

As noted in [28, Section 6.4.2], Theorems 1.1 and A.2 have the following consequences, which we also prove in the appendix. Recall that the symplectic Torelli group $\text{Symp}_h(M, \omega)$ is the subgroup of $\text{Symp}(M, \omega)$ that acts trivially on homology $H_*(M; \mathbb{Z})$.

Corollary 1.2. *Suppose (M^4, ω) is a symplectic rational manifold.*

- (1) *(Symplectic unknottedness of Lagrangian S^2) The symplectic Torelli group $\text{Symp}_h(M, \omega)$ acts transitively on homologous Lagrangian spheres.*
- (2) *(Smooth unknottedness of Lagrangian S^2) Homologous Lagrangian spheres are smoothly isotopic to each other.*

The smooth unknottedness for Lagrangian spheres was first noticed by Evans [11] in the case of del Pezzo surfaces, i.e. monotone rational symplectic manifolds.

Let (X_5, ω_0) be a monotone $\mathbb{CP}^2 \# 5\overline{\mathbb{CP}}^2$. By [28, Theorem 1.4] one can explicitly classify the homology classes $\xi \in H_2(X_5; \mathbb{Z})$ that can be represented by a Lagrangian sphere in (X_5, ω_0) . Furthermore Evans in [12, Theorem 1.3] computes the weak homotopy type of $\text{Symp}_h(X_5, \omega_0)$ and in [12, Section 6.1] shows $\pi_0(\text{Symp}_h(X_5, \omega_0))$ is a \mathbb{Z}_2 -quotient of the pure braid group $PBr(S^2, 5)$ on S^2 with 5 strands. These two results above together with Corollary 1.2 show that Lagrangian spheres in (X_5, ω_0) are unique up to Hamiltonian isotopy and a certain (explicit) braid group action.

Corollary 1.2 therefore allows for the first explicit description of Hamiltonian isotopy classes of Lagrangian spheres in a closed symplectic manifold where there is Lagrangian knotting. Hind [17] has done this in the non-compact setting with the plumbing of two T^*S^2 . In forthcoming work of the third author [44] this braid group action will be connected with Lagrangian Dehn twists along a finite set of Lagrangian spheres and the Hamiltonian isotopy classes of Lagrangian spheres in an A_n -singularity will be studied.

Such an explicit description brings up a more intriguing question. For a symplectic rational manifold Corollary 1.2(1) tells us we have a transitive group action of $\pi_0(\text{Symp}_h(M)) = \text{Symp}_h(M)/\text{Symp}_0(M)$ on $\mathcal{Lag}_\xi(M, S^2)$, the Hamiltonian isotopy classes of Lagrangian spheres in the class $\xi \in H_2(M; \mathbb{Z})$, and the stabilizer

of this action seems hard to understand. It may be possible to understand the stabilizer of the action of $\pi_0(\text{Symp}_h(M))$ on the Fukaya category in terms of braid group elements [20].

The unknotting picture for Lagrangian \mathbb{RP}^2 is more intriguing. We first have the following parallel results of Corollary 1.2 for small Betti numbers:

Theorem 1.3. *Let (M^4, ω) be a symplectic rational manifold and $b_2^-(M) \leq 8$.*

- (1) *$\text{Symp}_h(M, \omega)$ acts transitively on \mathbb{Z}_2 -homologous Lagrangian \mathbb{RP}^2 's.*
- (2) *\mathbb{Z}_2 -homologous Lagrangian \mathbb{RP}^2 's are smoothly isotopic.*

Note in Lemma 4.6 we prove there is a unique \mathbb{Z}_2 -homology class containing a Lagrangian \mathbb{RP}^2 when $b_2^-(M) \leq 2$.

While it is still possible that the uniqueness of \mathbb{Z}_2 -homologous \mathbb{RP}^2 's up to smooth isotopy is valid for an arbitrary symplectic rational manifold, the following result gives some hints about the complication of the problem:

Proposition 1.4. *Let $M = \mathbb{CP}^2 \# k \overline{\mathbb{CP}}^2$.*

- (1) *For $k \geq 9$ there exists a symplectic form ω on M with L a Lagrangian \mathbb{RP}^2 and S a symplectic (-1) -sphere, such that L and S have trivial \mathbb{Z}_2 -intersection, but L and S cannot be made disjoint with a smooth isotopy.*
- (2) *For $k \geq 10$ there exists L_0 and L_1 that are \mathbb{Z}_2 -homologous smoothly embedded \mathbb{RP}^2 's, which are not smoothly isotopic. Furthermore there are deformation equivalent symplectic forms ω_0 and ω_1 on M so that $L_i \subset (M, \omega_i)$ are Lagrangians.*

Our symplectic packing results also provide ways to construct disjoint Lagrangian S^2 's and \mathbb{RP}^2 's, and this leads to Floer-theoretic statements about properties of Lagrangians, in particular showing that they are not superheavy with respect to the fundamental class of quantum homology [10]. Let (X_k, ω_0) be a del Pezzo surface, i.e. a monotone $\mathbb{CP}^2 \# k \overline{\mathbb{CP}}^2$ with $0 \leq k \leq 8$.

Theorem 1.5. *Any Lagrangian $\mathbb{RP}^2 \subset (X_k, \omega_0)$ is not superheavy with respect to the fundamental quantum homology class $\mathbf{1} \in QH_4(X_k, \omega_0)$ when $k \geq 2$. Likewise for Lagrangian spheres $S^2 \subset X_k$ when $k \geq 3$.*

In Proposition 5.1 we build Lagrangian \mathbb{RP}^2 's in all X_k . We also note that this has an interesting application to detecting non-displaceable toric fibers in certain symplectic surfaces with “semi-toric type” structures (see Section 5.2).

The structure of the paper goes as follows. In Section 2 we recall the necessary tools from relative symplectic packing as well as the symplectic cutting procedure that lets us switch between Lagrangians and divisors. This allows us to prove Theorem 1.1. In Section 3 we study symplectic (-4) -spheres, which is related to Lagrangian \mathbb{RP}^2 's via symplectic cuts. In Section 4 we provide proofs of Theorem 1.3 and Proposition 1.4. In Section 5 we prove Theorem 1.5 and discuss the relation between symplectic packing and Floer-theoretic properties.

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2 Connectedness of ball packings

Given a symplectic manifold (M^{2n}, ω) the space of *symplectic ball packings* of M of size $\bar{\lambda} = (\lambda_1, \dots, \lambda_k)$ is the space of smooth embeddings

$$E_{\bar{\lambda}}(M, \omega) := \left\{ \phi : \prod_{i=1}^k B^{2n}(\lambda_i) \rightarrow M : \phi^* \omega = \omega_{std} \text{ and } \phi \text{ is injective} \right\}$$

where $B^{2n}(\lambda) = \{z \in \mathbb{C}^n : \pi |z|^2 \leq \lambda\}$ and $\omega_{std} = dx \wedge dy$. In [31, 32] McDuff established the connection between symplectic packing and the symplectic blow-up. Given a symplectic packing $\phi \in E_{\bar{\lambda}}(M, \omega)$ performing the symplectic blow-up results in the symplectic form ω_ϕ on $M \# k \overline{\mathbb{CP}}^n$ where $[\omega_\phi] = [\pi^* \omega] - \sum_{i=1}^k \lambda_i \text{PD}(E_i)$ where $\pi : M \# k \overline{\mathbb{CP}}^n \rightarrow M$ is the blow-down map and $\text{PD}(E_i)$ is the Poincare dual to the exceptional class E_i corresponding to the blow-up of M at the i -th ball. See [32, 37] for more details.

Lalonde and McDuff in [21, 22, 34] developed a method known as inflation which builds a symplectic deformation of (M, ω) by adding a two-form dual to a symplectic submanifold $C^2 \subset (M^4, \omega)$. In particular McDuff's [34, Theorem 1.2], and its generalization by Li-Liu [26, Proposition 4.11], proves if (M^4, ω) is a closed symplectic manifold with $b_2^+ = 1$, then any deformation between cohomologous symplectic forms is homotopic with fixed endpoints to an isotopy of symplectic forms. This homotopy is done by inflation along one parameter families of embedded holomorphic curves, whose existence is given by Taubes-Seiberg-Witten theory [42, 33]. Using the relation between symplectic packings and symplectic forms on the blow-up, McDuff [34, Corollary 1.5] used that deformation implies isotopy to prove the space of symplectic packings is connected when $b_2^+(M) = 1$.

2.1 In the complement of a symplectic submanifold

Consider now the relative setting, where $Z \subset (M^4, \omega)$ is a closed embedded symplectic surface and one considers symplectic packings $E_{\bar{\lambda}}(M \setminus Z, \omega)$ in the complement of Z . Biran [2, Lemma 2.1.A] worked out how to inflate a symplectic form

along certain symplectic surfaces $C \subset M$ that intersects Z positively so that Z stays a symplectic submanifold through the deformation, so extending the connectedness of ball packing just requires one to find the appropriate holomorphic curves to inflate along. When the Seiberg–Witten degree of Z is non-negative $d(Z) = c_1(Z) + Z^2 \geq 0$, then Z will be a J -holomorphic curve for regular almost complex structures on M so McDuff’s argument in [34, Corollary 1.5] generalizes immediately.

The situation is more delicate when $d(Z) < 0$, but recent work of McDuff [36] builds the appropriate curves in certain cases and leads to the following proposition, which is implicit in [36]. We have followed [34, Corollary 1.5] where McDuff proves the connectedness of ball packings in the absolute case when $b_2^+(M) = 1$. Note that two packings $\phi_0, \phi_1 \in E_{\bar{\lambda}}(M, \omega)$ are connected if and only if there is a symplectomorphism $F \in \text{Symp}_0(M, \omega)$ in the identity component of $\text{Symp}(M, \omega)$ such that $F \circ \phi_0 = \phi_1$.

Proposition 2.1. *Let (W^4, ω) be a closed rational or ruled symplectic 4-manifold and let $Z \subset W$ be a closed symplectic sphere, then the space of symplectic packing $E_{\bar{\lambda}}(W \setminus Z, \omega)$ is connected.*

Proof. Given two packings $\phi_0, \phi_1 : \coprod_{i=1}^k B^4(\lambda_i) \rightarrow (W \setminus Z, \omega)$ by applying an element of $\text{Symp}_0^c(W \setminus Z, \omega)$, the identity component of the group of compactly supported symplectomorphisms $\text{Symp}^c(W \setminus Z, \omega)$, we may assume that $\phi_0 = \phi_1$ as maps when restricted to $\coprod_{i=1}^k B^4(a\lambda_i)$ for $a > 0$ sufficiently small.

Let $\bar{\omega}_0$ and $\bar{\omega}_1$ be the symplectic forms on the blow-up $\bar{W} = W \# k\overline{\mathbb{CP}}^2$ associated to the ball packings ϕ_0 and ϕ_1 . Pick a deformation of symplectic forms $\bar{\omega}_t$ on \bar{W} as in the proof of [34, Corollary 1.5] such that $\bar{\omega}_t$ is constant in a neighborhood of Z and in $H^2(\bar{W}; \mathbb{R})$

$$[\bar{\omega}_t] = [\pi^*\omega] - \sum_{i=1}^d (\lambda_i - \rho_i(t)) \text{PD}(E_i)$$

where $\pi : \bar{W} \rightarrow W$ is the natural blow-down map, $\text{PD}(E_i) \in H^2(\bar{W}; \mathbb{Z})$ is Poincare dual to the exceptional class E_i associated to the exceptional divisor $e_i \subset \bar{W}$, and $\rho_i : [0, 1] \rightarrow [0, \lambda_i]$ are smooth functions equal to 0 in a neighborhood of $t = 0, 1$. By [36, Proposition 1.2.9] there is a compactly supported isotopy $\{\bar{F}_t\}_t$ in $\text{Diff}_0(\bar{W} \setminus Z)$ such that $\bar{F}_0 = \text{id}$ and $\bar{F}_1^* \bar{\omega}_1 = \bar{\omega}_0$.

From here the proof proceeds exactly as in [34, Corollary 1.5]. By blowing down \bar{F}_1 induces a symplectomorphism $F \in \text{Symp}^c(W \setminus Z, \omega)$ so that $F \circ \phi_0 = \phi_1$, so to finish the proof it suffices to show $F \in \text{Symp}_0^c(W \setminus Z, \omega)$. Note that the construction of $F = F^{(1)}$ can be done in a family $F^{(a)} \in \text{Symp}^c(W \setminus Z, \omega)$ by starting the construction with respect to ϕ_0 and ϕ_1 being restricted to the domain $\coprod_{i=1}^k B^4(a\lambda_i)$ for $a \in (0, 1]$. Since we assumed ϕ_0 and ϕ_1 are equal on sufficiently small balls, $F^{(a)} = \text{id}$ for a close to zero and hence $F \in \text{Symp}_0^c(W \setminus Z, \omega)$. \square

2.2 In the complement of a spherical Lagrangian

In this subsection we will prove Theorem 1.1. The simple but key observation is to translate to a relative setting with a symplectic sphere, which allows us to apply Proposition 2.1. This translation will be done using the following construction.

2.2.1 The symplectic cutting construction

Let (M^{2n}, ω) be a symplectic manifold and let $L \subset M$ be a closed Lagrangian admitting a metric with periodic geodesic flow. Let \mathcal{N} be a Weinstein neighborhood of $L \subset M$, then by using the periodic geodesic flow we can perform a symplectic cut [23] along $\partial\mathcal{N}$. This cut creates a pair of new symplectic manifolds $W_{\mathcal{N}}^+$ and $W_{\mathcal{N}}^-$ each having $Z_{\mathcal{N}}$ as a codimension 2 symplectic submanifold, where the Euler classes of $Z_{\mathcal{N}}$'s normal bundles in $W_{\mathcal{N}}^{\pm}$ satisfy $e(\nu^+) = -e(\nu^-) \in H^2(Z_{\mathcal{N}})$. The manifolds are

$$W_{\mathcal{N}}^+ = (M \setminus \text{Int } \mathcal{N}) / \sim \quad W_{\mathcal{N}}^- = \mathcal{N} / \sim \quad Z_{\mathcal{N}} = \partial\mathcal{N} / \sim$$

where the equivalence relations are given by identifying point on $\partial\mathcal{N}$ that lie on the same geodesic, and the symplectic manifold M can be recovered by taking the symplectic sum [14] of $W_{\mathcal{N}}^{\pm}$ along $Z_{\mathcal{N}}$

$$M = W_{\mathcal{N}}^+ \#_{Z_{\mathcal{N}}} W_{\mathcal{N}}^-. \quad (2.1)$$

Remark 2.2. *Let us record what arises when L is a Lagrangian S^2 or \mathbb{RP}^2 .*

- (1) *Lagrangian S^2 : We have $W^- = (S^2 \times S^2, \sigma \oplus \sigma)$ with $L \subset W^-$ being the anti-diagonal Lagrangian sphere, where $Z \subset W^-$ is the diagonal symplectic sphere and $Z \subset W^+$ is a (-2) symplectic sphere.*
- (2) *Lagrangian \mathbb{RP}^2 : We have $W^- = \mathbb{CP}^2$ with $L \subset W^-$ being the standard \mathbb{RP}^2 , where $Z \subset W^-$ is the quadric $Q = \{z_0^2 + z_1^2 + z_2^2 = 0\}$ and $Z \subset W^+$ is a (-4) symplectic sphere.*

For more details and other examples see [1].

In [15, Theorem 1.1] Hausmann–Knutson determined the effect of symplectic cutting in general on the rational cohomology ring and it leads to the following lemma. Note that [15, Theorem 1.1] assumes the symplectic cut is by a global Hamiltonian S^1 -action, which need not be true in our case. However since $H^*(M; \mathbb{Q}) \rightarrow H^*(\mathcal{N}; \mathbb{Q})$ is surjective if $\mathcal{N} \subset M$ is a Weinstein neighborhood for a Lagrangian S^2 or \mathbb{RP}^2 , the proof of [15, Theorem 1.1] still applies in our case.

Lemma 2.3. *Let (M^4, ω) be a closed symplectic manifold with $L \subset M$ a Lagrangian S^2 or \mathbb{RP}^2 and let $W_{\mathcal{N}}^+$ be the symplectic manifold built by cutting out a Weinstein neighborhood \mathcal{N} of L , then we have the following:*

- (1) $b_2^+(W_{\mathcal{N}}^+) = b_2^+(M)$.

(2) If $L = S^2$, then $b_2^-(W_{\mathcal{N}}^+) = b_2^-(M)$. If $L = \mathbb{RP}^2$, then $b_2^-(W_{\mathcal{N}}^+) = b_2^-(M) + 1$.

(3) If M is rational or ruled, then $W_{\mathcal{N}}^+$ is rational or ruled respectively.

Proof. For (1) and (2): If L is a Lagrangian S^2 , then $H^*(W_{\mathcal{N}}^+; \mathbb{Q}) \cong H^*(M; \mathbb{Q})$ as rings by [15]. If L is a Lagrangian \mathbb{RP}^2 , then by [15] the intersection forms are related by

$$Q_{W_{\mathcal{N}}^+} = Q_{\overline{\mathbb{CP}}^2} \oplus Q_M = (-1) \oplus Q_M \quad (2.2)$$

where $PD(Z_{\mathcal{N}}) \in H^2(W_{\mathcal{N}}^+; \mathbb{Q})$ is identified with $2h \in H^2(\overline{\mathbb{CP}}^2; \mathbb{Q})$.

For (3): Let κ denote the symplectic Kodaira dimension, then by [6, Theorem 1.1] and (2.1) it follows that $\kappa(W_{\mathcal{N}}^+) \leq \kappa(M)$. For closed 4-dimensional symplectic manifolds having $\kappa = -\infty$ is equivalent to being rational or ruled [29]. Since [15] gives $H_1(W_{\mathcal{N}}^+; \mathbb{Q}) \cong H_1(M; \mathbb{Q})$ it follows that $W_{\mathcal{N}}^+$ is rational if M is rational, and likewise for ruled. \square

In the case of a Lagrangian sphere, Lemma 2.3 can also be proved by noting that $W_{\mathcal{N}}^+$ is a symplectic deformation of M .

2.2.2 Proving Theorem 1.1

In the setting of the symplectic cutting construction, by design $W_{\mathcal{N}}^+ \setminus Z_{\mathcal{N}}$ is symplectomorphic to $M \setminus \mathcal{N}$. Therefore we immediately have the following observation on symplectic packings of $M \setminus L$.

Lemma 2.4. *The space of ball packings $E_{\bar{\lambda}}(M \setminus L)$ in $M \setminus L$ is connected if and only if the space of ball packings $E_{\bar{\lambda}}(W_{\mathcal{N}}^+ \setminus Z_{\mathcal{N}})$ in $W_{\mathcal{N}}^+ \setminus Z_{\mathcal{N}}$ is connected for all sufficiently small Weinstein neighborhoods \mathcal{N} of $L \subset M$.*

This in turn leads to the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $(W_{\mathcal{N}}^+, \omega)$ be the result of cutting out a Weinstein neighborhood \mathcal{N} of L in M , with $Z_{\mathcal{N}} \subset W_{\mathcal{N}}^+$ being the resulting symplectic sphere. By Lemma 2.3 we have that $W_{\mathcal{N}}^+$ is rational or ruled, so by Proposition 2.1 the space of ball packings in $W_{\mathcal{N}}^+ \setminus Z_{\mathcal{N}}$ is connected. The theorem now follows from Lemma 2.4. \square

3 Symplectic (-4) -spheres in rational manifolds

In this section we provide a classification of a special type of classes which can be represented by symplectic (-4) -spheres in symplectic rational manifold (W^4, ω) . This will be crucial to our study of Lagrangian \mathbb{RP}^2 's due to the symplectic cutting construction.

We first briefly establish some notation. If $M = S^2 \times S^2$, then we will let $A, B \in H_2(M; \mathbb{Z})$ be the homology classes for each factor. If $M = \mathbb{CP}^2 \# k \overline{\mathbb{CP}}^2$, we

will represent its second homology classes by the basis $\{H, E_1, \dots, E_k\}$, where H is the line class and E_i are orthogonal exceptional classes. Denote the class

$$K_0 = -3H + E_1 + \dots + E_k$$

and by [26, Theorem 1] we may always assume that this is the canonical class for symplectic rational manifolds. Define the K_0 -exceptional classes as the spherical classes C satisfying $K_0 \cdot C = -1$, and by [26] these classes must be represented by an symplectic exceptional sphere.

There is a special kind of transformation on the second homology group of rational manifolds called the *Cremona transform*, which are reflections

$$A \mapsto A + (A \cdot L_{ijk}) L_{ijk} \quad \text{where} \quad L_{ijk} = H - E_i - E_j - E_k \text{ for } i > j > k$$

that preserve the class K_0 . When $k \leq 3$, we also include the reflection with respect to $L_{ij} = E_i - E_j$. Cremona transformations can always be realized by diffeomorphisms [24] (explicitly this can be done by a smooth version of Seidel's Dehn twist). See [38, 28] for more detailed discussions on these transformations. If two classes are connected by a series of Cremona transforms, we say they are *Cremona equivalent*.

Let $Z \subset W$ be a symplectic (-4) -sphere, then the *rational blow-down along Z* is the symplectic manifold built by performing a symplectic sum [14] of W along Z with the standard quadric Q in \mathbb{CP}^2 and is denoted $M = W_Z \#_Q \mathbb{CP}^2$. This operation is the inverse of the symplectic cutting construction from Section 2.2 in the case of a Lagrangian $\mathbb{RP}^2 \subset M$.

Remark 3.1. *Unlike the symplectic cutting operation that keeps the manifold rational or ruled in our case by Lemma 2.3, the same is not true when we blow-down (-4) symplectic spheres. For example the class $2K_0 = 6H - 2(E_1 + \dots + E_{10})$ is represented by an embedded (-4) -symplectic sphere for an appropriate symplectic form on $\mathbb{CP}^2 \# 10\overline{\mathbb{CP}}^2$, but the symplectic blow-down is not a rational manifold [7, Section 4.2].*

Proposition 3.2. *Let (W^4, ω) be a symplectic rational manifold and $Z \subset W$ be an embedded symplectic (-4) -sphere. If $W = S^2 \times S^2$, then $[Z] = A - 2B$ or $B - 2A$. If $W = \mathbb{CP}^2 \# k\overline{\mathbb{CP}}^2$, then $k \geq 2$ and furthermore $[Z] \in H_2(W, \mathbb{Z})$ is Cremona equivalent to $-H + 2E_1 - E_2$ provided the rational blow-down M of W along Z is a rational manifold.*

Proof. A direct computation with the adjunction formula implies all but the case where $W = \mathbb{CP}^2 \# k\overline{\mathbb{CP}}^2$ with $k \geq 4$. From the relation between rational blow-down and symplectic cutting from Section 2.2, it follows from equality (2.2) that $b_2^-(W) = 1 + b_2^-(M)$. It follows that M is not minimal since it is rational and $b_2^-(M) = k - 1 \geq 3$.

By [7, Theorem 1.2] it follows that either

- (i) there exists a symplectic exceptional sphere disjoint from Z , or
- (ii) there exist two disjoint symplectic exceptional spheres C_1 and C_2 each intersecting Z exactly once and positively.

Therefore we can assume after blowing down some number of exceptional spheres disjoint from Z that the resulting manifold \overline{W} is minimal or (ii) occurs for $Z \subset \overline{W}$.

If we reach a minimal manifold then $\overline{W} = S^2 \times S^2$, since the other possibility, \mathbb{CP}^2 , does not have a symplectic (-4) -curve. So $[Z]$ is either $A - 2B$ or $B - 2A$ in $H_2(\overline{W}; \mathbb{Z})$ and since we needed to have blow-down at least 3 exceptional spheres in W to reach \overline{W} , by blowing back up to W it follows that the classes $A - 2B$ and $B - 2A$ can be represented in the form of part (ii) in our assertion, by an appropriate change of basis from classes A, B to the standard basis consisting of $\{H, E_1, \dots, E_k\}$.

Assume now (ii) occurs for $Z \subset W$, where W has no exceptional spheres disjoint from Z . From the classification of exceptional classes in rational manifolds up to Cremona equivalence [38, Proposition 1.2.12], [28, Corollary 4.5], any exceptional class is Cremona equivalent to E_i for some i . Also since Cremona transformations can be realized by diffeomorphisms, we may now assume without loss of generality $[C_i] = E_i$ for $i = 1, 2$.

Taking our divisor Z into account, the effect of blowing down both C_i can be described as performing symplectic fiber sums [14, Theorem 1.4] of (W, Z) with (\mathbb{CP}^2, ℓ) along C_i and ℓ_i , where ℓ is a line intersecting the line ℓ_i transversally, for $i = 1$ and 2 . The result is (W', Z') where Z' is a symplectic (-2) -sphere in a rational symplectic manifold W' and under $\iota : H_2(W') \rightarrow H_2(W)$ the canonical inclusion we have $\iota[Z'] = [Z] - E_1 - E_2$.

It follows from [28, Proposition 4.10] that $[Z'] \in H_2(W')$ is Cremona equivalent to the class $E_3 - E_4$. Since Cremona transforms are given by reflections about homology classes, the inclusion ι translates the Cremona equivalent of $[Z']$ and $E_3 - E_4$ in $H_2(W')$ to a series of Cremona transforms of $H_2(W)$ that fix E_1 and E_2 while sending $\iota[Z']$ to the class $E_3 - E_4 - E_1 - E_2$. This class is equivalent to $-H + 2E_3 - E_4$ by reflection with respect to $H - E_1 - E_2 - E_3$, and $-H + 2E_3 - E_4$ is equivalent to the form in the assertion. \square

Corollary 3.3. *If (W, ω) is a symplectic rational manifold and Z is an embedded symplectic (-4) -sphere, then in the complement of Z there are $n = b_2^-(W) - 1$ disjoint exceptional symplectic spheres $\{D_i\}_{i=1}^n$. Moreover, given any set of orthogonal exceptional classes pairing trivially with $[Z]$, there are disjoint exceptional spheres in $W \setminus Z$ representing these classes.*

Proof. There is nothing to prove if $W = S^2 \times S^2$ or $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$, so by Proposition 3.2 we can assume $W = \mathbb{CP}^2 \# (n+1)\overline{\mathbb{CP}}^2$ with $n \geq 1$ and $[Z] = -H + 2E_1 - E_2$. In this case,

$$E'_1 = H - E_1 - E_2 \quad \text{and} \quad E'_i = E_{i+1} \quad \text{for } i = 2, \dots, n.$$

is a set of n orthogonal exceptional classes that pair trivially with $[Z]$. Now it suffices to prove the second part, which follows from McDuff's result [36, Proposition 1.2.5] that asserts we can find a compatible almost complex structure J on (W, ω) for which Z is a complex submanifold and the classes E'_i are represented by a J -holomorphic embedded spheres D_i . By positivity of intersections it follows that the D_i are disjoint from each other and Z . \square

Definition 3.4. If (W^4, ω) is a symplectic rational manifold and $Z \subset W$ is a symplectic (-4) -sphere, then a set of $n = b_2^-(W) - 1$ disjoint exceptional spheres $\{D_i\}_{i=1}^n$ in $W \setminus Z$ is called a *push off set of Z* and the set $\{[D_i]\}_{i=1}^n$ of exceptional classes is called a *push off system of Z* .

4 Lagrangian \mathbb{RP}^2 's in a rational manifolds

In this section (M^4, ω) will always be a symplectic $\mathbb{CP}^2 \# k\overline{\mathbb{CP}}^2$ where the canonical class is $K_0 = -3H + E_1 + \dots + E_k$ and $L \subset M$ will be a Lagrangian \mathbb{RP}^2 . Note that $S^2 \times S^2$ does not contain Lagrangian \mathbb{RP}^2 since it does not have a \mathbb{Z}_2 -homology class with non-trivial self-pairing, so we need not put it into consideration. We will let $(W, Z) = (W_{\mathcal{N}}^+, Z_{\mathcal{N}})$ be the result of performing the symplectic cutting procedure from Section 2.2 on (M, L) . Note by construction that $Z \subset W$ is a symplectic (-4) sphere and W is a symplectic rational manifold $\mathbb{CP}^2 \# (k+1)\overline{\mathbb{CP}}^2$ by Lemma 2.3.

4.1 Push off and orthogonal systems

Translating our knowledge about symplectic (-4) spheres leads to the following result.

Proposition 4.1. *If (M^4, ω) is a symplectic rational manifold with $L \subset M$ a Lagrangian \mathbb{RP}^2 , then there is a set of $k = b_2^-(M)$ disjoint exceptional spheres $\{D_i\}_{i=1}^k$ in $M \setminus L$.*

Proof. By construction $W \setminus Z$ is symplectomorphic to a subset of $M \setminus L$ and by Corollary 3.3 there are $k = b_2^-(W) - 1 = b_2^-(M)$ disjoint exceptional spheres $\{D_i\}_{i=1}^k$ in $W \setminus Z$. \square

In light of Proposition 4.1, we introduce the following analogue of Definition 3.4.

Definition 4.2. If (M^4, ω) is a symplectic rational manifold and $L \subset M$ is a Lagrangian \mathbb{RP}^2 , then a set $k = b_2^-(M)$ disjoint exceptional spheres $\{D_i\}_{i=1}^k$ in $M \setminus L$ is called a *push off set of L* and the set $\{[D_i]\}_{i=1}^k$ of exceptional classes is called a *push off system of L* .

The following proposition shows the importance of the notion of a push off system in proving Theorem 1.3.

Proposition 4.3. *Suppose two Lagrangian \mathbb{RP}^2 's in a symplectic rational manifold have a common push off system, then they are both (1) smoothly isotopic and (2) Torelli symplectomorphic.*

Proof. For the first claim, Corollary 3.3 and [28, Proposition 3.4] ensures that up to symplectic isotopy the Lagrangian L_i share common a push off set. By blowing down the push-off set we have Lagrangian \mathbb{RP}^2 's in \mathbb{CP}^2 , which are unique up to Hamiltonian isotopy [17, 28]. This can be lifted to a smooth isotopy of M since avoiding a blow-up region along an isotopy is equivalent to avoiding a point in the smooth category. See also the argument in the proof of [28, Theorem 1.6]. The proof of the second claim is identical to the proof of Corollary 1.2 in Appendix A.2. \square

The rest of the section lists some homological restrictions on Lagrangian \mathbb{RP}^2 classes, which will prove useful. In particular, the following definition, albeit purely topological, will provide more accurate homological information regarding a Lagrangian \mathbb{RP}^2 class.

Definition 4.4. Let (M, ω) be a symplectic $\mathbb{CP}^2 \# k\overline{\mathbb{CP}}^2$ and $A \in H_2(M; \mathbb{Z}_2)$, then a \mathbb{Z}_2 -orthogonal system for A is a set $\{F_i\}_{i=1}^k$ of pairwise orthogonal exceptional classes in $H_2(M; \mathbb{Z})$ such that $A \cdot F_i = 0$ in \mathbb{Z}_2 -homology for all i . If A admits a \mathbb{Z}_2 -orthogonal system, then A is called a K_0 -Lagrangian \mathbb{RP}^2 class.

Remark 4.5. *Note that a \mathbb{Z}_2 -orthogonal system consists of only homological conditions for the exceptional classes, while a push off system (Definition 4.2) requires exceptional representatives disjoint from L . We have that*

$$\{\text{push of systems of } L\} \subset \{\mathbb{Z}_2\text{-orthogonal systems of } [L]\}$$

as collections of subsets of $H_2(M; \mathbb{Z})$.

Lemma 4.6. *Let (M, ω) be a symplectic $\mathbb{CP}^2 \# k\overline{\mathbb{CP}}^2$.*

- (1) *If $L \subset M$ is a Lagrangian \mathbb{RP}^2 , then $[L] \in H_2(M; \mathbb{Z}_2)$ is a K_0 -Lagrangian \mathbb{RP}^2 class.*
- (2) *Each K_0 -Lagrangian \mathbb{RP}^2 class is Cremona equivalent to the \mathbb{Z}_2 -reduction H .*
- (3) *The \mathbb{Z}_2 -reduction $H \in H_2(M; \mathbb{Z}_2)$ is the unique K_0 -Lagrangian \mathbb{RP}^2 class when $k \leq 2$.*
- (4) *Every K_0 -Lagrangian \mathbb{RP}^2 class is represented by a smoothly embedded \mathbb{RP}^2 , which is Lagrangian for some symplectic form on M .*

Proof. Part (1) follows directly from Proposition 4.1 and definitions.

To prove (2), we need to show the following statement: given k orthogonal exceptional classes, there is a Cremona transform sending this set to $\{E_1, \dots, E_k\}$. This is proved in [28, Corollary 4.5] for a set of at most $k - 2$ exceptional classes

(and does not hold in general for a set of $k-1$ exceptional classes, e.g. when $k=2$ and the set is $\{H - E_1 - E_2\}$).

Apply the proved case for $k-2$ exceptional classes to $\{E_3, \dots, E_k\}$. This reduces the problem to the case of $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}}^2$. But the only orthogonal set of exceptional classes containing 2 elements is $\{E_1, E_2\}$. This means the induction necessarily terminates in this case and the resulting Cremona transform sends the set of exceptional classes to $\{E_i\}_{i=1}^{b^-}$ as desired. It is then clear that the only non-trivial \mathbb{Z}_2 -class orthogonal to this set is the \mathbb{Z}_2 -reduction of H .

For part (3), when $k \leq 2$ there is a unique set of k orthogonal exceptional classes, so the only K_0 -Lagrangian \mathbb{RP}^2 class is the reduction of H .

It is easy to make the proper transform of the standard $\mathbb{RP}^2 \subset \mathbb{CP}^2$ Lagrangian, just by packing sufficiently small balls in the complement. Therefore part (4) follows from (2) since Cremona transforms are homological actions of smooth Dehn twists. \square

Return now to the case where $L \subset M = \mathbb{CP}^2 \# k\overline{\mathbb{CP}}^2$ is a Lagrangian \mathbb{RP}^2 and (W, Z) is the result of performing the symplectic cutting procedure. The next lemma describes the relationship between the push off systems of $L \subset M$ and $Z \subset W$.

Lemma 4.7. *The isomorphism $H_2(W \setminus Z; \mathbb{Z}) \rightarrow H_2(M \setminus L; \mathbb{Z})$ induces a bijection:*

$$\tau : \{\text{push off systems of } Z\} \rightarrow \{\text{push off systems of } L\}$$

Proof. The map τ is defined as follows, a push off set of Z survives the rational blow-down along Z and hence defines a push off set of L . The homology isomorphism ensures τ is well-defined. Since the map $H_2(M \setminus L; \mathbb{Z}) \rightarrow H_2(M; \mathbb{Z})$ is injective by the long exact sequence for the pair $(M, M \setminus L)$, and there is an isomorphism $H_2(W \setminus Z; \mathbb{Z}) \cong H_2(M \setminus L; \mathbb{Z})$, it follows that τ is injective.

For surjectivity we argue as follows. Given a push off system of L , perform a symplectic cut on a sufficiently small Weinstein neighborhood \mathcal{N}' of L to get a push off system for Z' in an ambient symplectic manifold W' . Since (W, Z) and (W', Z') differ by a symplectic deformation near the divisors and exceptional classes are invariant under symplectic deformation, Corollary 3.3 builds a push off set for Z that τ maps to the original push off system of L . \square

4.1.1 Proof of Theorem 1.3

Lemma 4.8. *Suppose L is a Lagrangian \mathbb{RP}^2 in $M = \mathbb{CP}^2 \# k\overline{\mathbb{CP}}^2$ for $k \leq 8$, then each \mathbb{Z}_2 -orthogonal system of $[L]$ is a push off system of L .*

Proof. Note there is nothing to prove if $k=0$, and so if we let (W, Z) be the result of doing the symplectic cut to (M, L) , by Lemma 2.3 we may assume that $W = \mathbb{CP}^2 \# (k+1)\overline{\mathbb{CP}}^2$ where $1 \leq k \leq 8$.

When $k \leq 8$, there are only finitely many exceptional classes and hence a finite number of push off systems and \mathbb{Z}_2 -orthogonal systems. Hence by the

inclusion in Remark 4.5 it suffices to perform a count of each of these sets. As we will now explain this reduces to a count of certain homology classes, and this count is performed in Appendix B.

By Lemma 4.6 we can assume that $[L] = H \in H_2(M; \mathbb{Z}_2)$, so we only need to count \mathbb{Z}_2 -orthogonal systems for $H \in H_2(M; \mathbb{Z}_2)$. This is equivalent to counting sets of k orthogonal exceptional classes in $H_2(M; \mathbb{Z})$ that are \mathbb{Z}_2 -orthogonal to H .

By Lemma 4.7 push off systems of L in M correspond to push off systems of Z in W and by Proposition 3.2 we can assume that $[Z] = S = -H + 2E_1 - E_2$. So by the second part of Corollary 3.3 it suffices to count sets of k orthogonal exceptional classes in $H_2(W; \mathbb{Z})$ that pair trivially with S in $H_2(W; \mathbb{Z})$. \square

We can now prove Theorem 1.3.

Proof of Theorem 1.3. If L_1 and L_2 are in the same \mathbb{Z}_2 -homology class, then they have a common \mathbb{Z}_2 -orthogonal system and hence by Lemma 4.8 they share a common push off system since $k \leq 8$. Therefore by Proposition 4.3 they are smoothly isotopic and Torelli symplectomorphic. \square

4.2 Homology correspondence

We now attempt to understand the relation between \mathbb{Z}_2 -orthogonal classes and push-off systems of Lagrangian $\mathbb{R}P^2$ in more detail. This leads to the failure of Lemma 4.8 when $k \geq 9$ thus a construction for twisted Lagrangian $\mathbb{R}P^2$ as in Proposition 1.4.

4.2.1 Associated basis

Here we will explicitly describe the homological effect of rational blow-down in rational manifolds for $M = W_Z \#_Q \mathbb{C}P^2$.

Lemma 4.9. (1) *There is an orthogonal basis $\{H', E'_0, E'_1, \dots, E'_k\}$ of $H_2(W; \mathbb{Z})$, where H' is a line class and the E'_i are exceptional classes. Elements of this basis are represented by disjoint symplectic spheres $\{h', e'_0, \dots, e'_k\}$. The symplectic spheres can be picked so each of e'_1, e'_2, e'_3 intersect Z exactly once and positively, while h', e'_4, \dots, e'_k are disjoint from Z .*

(2) *There is an orthogonal basis $\{H, U_1, U_2, U_3, E_4, \dots, E_k\}$ for $H_2(M; \mathbb{Z})$, where H is a line class and the rest are exceptional classes. Elements of this basis are represented by disjoint symplectic spheres $\{h, u_1, u_2, u_3, e_4, \dots, e_k\}$. Furthermore $[L] = U_1 + U_2 + U_3 \in H_2(M; \mathbb{Z}_2)$.*

Proof. By Proposition 3.2 we may assume that $[Z] = E'_0 - E'_1 - E'_2 - E'_3$, since it is Cremona equivalent to $-H' + 2E'_0 - E'_1$, and from here [36, Proposition 1.2.7] builds the appropriate spheres h' and e'_i for $i = 1, \dots, k$ for generic almost complex structures for which Z is holomorphic. The symplectic representative of e'_0 will not be used later, but one can always attain a desired representative by perturbing

the almost complex structure from the previous step to a generic one. This proves part (1).

For part (2) since the curves h', e'_4, \dots, e'_k in W are disjoint from Z they can also be viewed as curves in $M = W_Z \#_Q \mathbb{CP}^2$ after performing the symplectic sum. So we are left to build the curves u_1, u_2, u_3 in M . To build u_1 the gluing method for relative Gromov–Witten invariants [27, 18] allow us to glue the curves e'_1 and e'_2 in W , which intersect Z transversally, to a degree one curve e'_{12} in \mathbb{CP}^2 that intersects the quadric twice at the points x_1 and x_2 where $x_i = e'_i \cap Z$. Denote the resulting holomorphic curve u_1 and likewise build the holomorphic curves u_2 and u_3 . By construction we have that in \mathbb{Z}_2 -homology $[L] \cap U_i = 1$ and $[L] \cap H = [L] \cap E_i = 0$, and this is enough to determine that $[L] = U_1 + U_2 + U_3$. \square

4.2.2 The monomorphism ι

Let $\mathcal{L}_L \subset H_2(M, \mathbb{Z})$ be the subgroup formed by elements $[\alpha]$ with trivial \mathbb{Z}_2 -intersection product with $[L] \in H_2(M; \mathbb{Z}_2)$. In Lemma 4.11 we will give an explicit monomorphism $\iota : \mathcal{L}_L \hookrightarrow H_2(W; \mathbb{Z})$, but first we need the following lemma whose proof was pointed out to us by Robert Gompf.

Lemma 4.10. *Let X^4 be a closed oriented manifold with L an embedded \mathbb{RP}^2 . If a homology class $[\alpha] \in H_2(X; \mathbb{Z})$ pairs trivially $[\alpha] \cdot [L] = 0$ with $[L]$ in \mathbb{Z}_2 -homology, then $[\alpha]$ has a representative cycle α such that $\alpha \cap L = \emptyset$.*

Proof. Pick an orientable representative cycle C of $[\alpha]$ intersecting L transversely such that the number of total intersection points is even. One may cancel a pair of the intersections x_1, x_2 in the following way: By fixing a local orientation on the interior of the unique 2-cell of \mathbb{RP}^2 , one can define local intersection numbers ± 1 at x_1 and x_2 . Now pick a path γ in \mathbb{RP}^2 connecting x_1 and x_2 that either reserves or preserves the local orientation, depending on if the local intersection numbers match or not. Form a new cycle now by removing small neighborhoods of x_1 and x_2 in C and then connecting the boundaries with a tube $S^1 \times [0, 1]$ corresponding to a small circle section over the path γ in the normal bundle of $\mathbb{RP}^2 \subset X$. Since X^4 is orientable, flipping the local orientation in \mathbb{RP}^2 is equivalent to flipping the normal orientation and hence we have a new orientable cycle representing $[\alpha]$ with two fewer intersections with L . \square

Lemma 4.11. *Suppose $M = \mathbb{CP}^2 \# k \overline{\mathbb{CP}}^2$ with $k \geq 3$, then there is a monomorphism $\iota : \mathcal{L}_L \hookrightarrow H_2(W; \mathbb{Z})$ with the following properties:*

- (1) *It preserves the \mathbb{Z} -intersection product.*
- (2) *In the bases from Lemma 4.9 if $[\alpha] = aH - t_1U_1 - t_2U_2 - t_3U_3$ then*

$$\iota([\alpha]) = aH' - \frac{-t_1+t_2+t_3}{2} E'_0 - \frac{t_1-t_2+t_3}{2} E'_1 - \frac{t_1+t_2-t_3}{2} E'_2 - \frac{t_1+t_2+t_3}{2} E'_3 \quad (4.1)$$

and $\iota(E_i) = E'_i$ for $i \geq 4$.

- (3) If $E \in H_2(M; \mathbb{Z})$ is an exceptional class, then E has an embedded symplectic spherical representative disjoint from L for some K_0 -symplectic form if and only if $E \in \mathcal{L}_L$ and $\iota(E)$ is also an exceptional class.

Proof. Given $[\alpha] \in \mathcal{L}_L$ define $\iota([\alpha]) \in H_2(W; \mathbb{Z})$ by picking a representative cycle α in $M \setminus L$ and view it as a cycle in W . Since $H_2(M \setminus L) \rightarrow H_2(M)$ is injective, it follows that this map is well-defined. It is clear that this map preserves the intersection pairing.

Part (2) is verified by comparing intersection numbers with the associated basis of the cycle α . If $[\alpha] = aH - t_1U_1 - t_2U_2 - t_3U_3$ and $\iota([\alpha]) = aH - \sum_{i \geq 0} b'_i E'_i$, then $b_i = b'_i$, $b'_0 = b'_1 + b'_2 + b'_3$, $b'_2 + b'_3 = t_1$, $b'_1 + b'_3 = t_2$, $b'_1 + b'_2 = t_3$ from construction of u_1 , u_2 , and u_3 in Lemma 4.9. This is exactly the form given by the statement of the lemma after elementary computation.

For part (3), the ‘only if’ part is trivial. We prove the ‘if’ part, the definition of ι implies that $\langle \iota(E), [Z] \rangle = 0$, so by [36, Theorem 1.2.1], we have a representative of symplectic exceptional sphere in W disjoint from Z . This is also a symplectic exceptional sphere in $M \setminus L$ that represents E . \square

4.3 Proof of Proposition 1.4

4.3.1 A Lagrangian \mathbb{RP}^2 in $B_3 = B \# 3\overline{\mathbb{CP}}^2$

Lemma 4.12. *For a symplectic ball blown-up three times ($B_3 = B \# 3\overline{\mathbb{CP}}^2, \omega$) if the exceptional classes have equal small symplectic area, then B_3 admits L an embedded Lagrangian \mathbb{RP}^2 such that $[L] = E_1 + E_2 + E_3 \in H_2(B_3; \mathbb{Z}_2)$.*

Proof. Suppose the three blow-ups are of size α , we are going to obtain B_3 in a different way as follows. We perform the following blow-up on B instead: blow-up once of size $t = \frac{3\alpha}{2} + \epsilon$, where $0 < \epsilon \ll \alpha$. We denote this resulting exceptional sphere as e_0 and its class as E_0 . Again perform three more blow-ups on e_0 with equal size $\frac{\alpha}{2} - \epsilon$, where the resulting exceptional spheres and classes are denoted as e_i and E_i , respectively, for $i = 1, 2, 3$. We now have a symplectic sphere e_{0123} in class $E_0 - E_1 - E_2 - E_3$ of self-intersection (-4) and area 4ϵ in $(B_4 = B \# 4\overline{\mathbb{CP}}^2, \omega')$. The gluing construction along e_{0123} with \mathbb{CP}^2 along a quadric shows that the resulting symplectic open manifold is B_3 with the correct symplectic areas. One way of verification is that, one can regard all these surgeries being supported in the complement of a line in \mathbb{CP}^2 , which gives $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ by [6, Theorem 1.1] and [15, Theorem 1.1]. But removing a line from $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ clearly gives B_3 . The symplectic areas are checked directly from the construction in Lemma 4.9. For example, consider $\omega(u_1)$. It consists of the gluing of e_1 , e_2 and e_{12} . Notice e_{12} is a line in the \mathbb{CP}^2 used for rational blow-down. Therefore, it has half the area of the quadric, which has equal area of e_{0123} . It follows that

$$\omega(u_1) = \omega'(e_1) + \omega'(e_2) + \omega'(e_{12}) = (\frac{\alpha}{2} - \epsilon) + (\frac{\alpha}{2} - \epsilon) + \frac{1}{2}4\epsilon = \alpha$$

so the exceptional sphere has the correct area. \square

Remark 4.13. Notice that from the above construction, the resulting Lagrangian \mathbb{RP}^2 has the three exceptional spheres as its associated basis from Lemma 4.9.

4.3.2 Proof of Proposition 1.4

Proof of Proposition 1.4. We only need to prove this for $k = 9$. For larger k the argument carries over with no further efforts. For part (1) by Lemma 4.12 and Remark 4.13 we can construct a Lagrangian L which is an embedded \mathbb{RP}^2 in certain $(M = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2, \omega')$ such that its associated basis from Lemma 4.9 is $\{H, U_1, U_2, U_3, E_4, \dots, E_9\}$ and its \mathbb{Z}_2 -homology class is $[L] = U_1 + U_2 + U_3$.

Since this can be done such that $U_1, U_2, U_3, E_4, \dots, E_9$ have ω' -area as small as we like by the construction in Lemma 4.12, we may assume that

$$E = 3H - 2U_1 - E_4 - \dots - E_9$$

is represented by an exceptional symplectic sphere in (M, ω') . Note that $E \cdot [L] = 0$ in \mathbb{Z}_2 -homology, so we can apply part (2) in Lemma 4.11 to see that

$$\iota(E) = 3H' + E'_0 - E'_1 - E'_2 - E'_3 - E'_4 - \dots - E'_9 \in H_2(W; \mathbb{Z}).$$

It is well-known this class has minimal symplectic genus equal 1, which goes back to Kervaire–Milnor [19], see also [25, (3.1) on pp. 3]. Therefore since $\iota(E)$ is not an exceptional class, it follows from part (3) in Lemma 4.11 that E does not have a symplectic representative disjoint from L .

For part (2) consider $(M' = M \# \overline{\mathbb{CP}}^2, \omega)$, which is a symplectic blow-up of M away from L . The new exceptional class and sphere will be denoted as E_{10} and e_{10} , respectively. Note that $e_{10} \cap L = \emptyset$. Now extend $\{E, E_{10}\}$ to a homology basis consisting of a line class and exceptional classes otherwise. This can always be done, for example, by taking the Cremona transforms which sends E_1 to E and fixes E_{10} (such Cremona transforms can always be found, see [38]) and act on the original basis $\{H, E_1 \dots E_{10}\}$.

Let F be a diffeomorphism $F : M' \rightarrow M'$ inducing the Cremona transform that is reflection along $E - E_{10}$, then F_* switches E and E_{10} and acts trivially on the rest of basis elements. Clearly $L' = F(L) \hookrightarrow (M', (F^{-1})^*\omega)$ is a Lagrangian embedding disjoint from the spherical representative $F(e_{10})$ of class E . By the form of F_* and the fact that $[L] \cdot E = [L] \cdot E_{10} = 0$ in \mathbb{Z}_2 -pairing, we see that L' is \mathbb{Z}_2 -homologous to L .

To see that L and L' are not smoothly isotopic, we assume the contrary. Such an isotopy can be extended to a family of diffeomorphism f_t for $t \in [0, 1]$ of M' . Let e be a representative of E with $e \cap L' = \emptyset$, for example, $e = F(e_{10})$. Then we have that $f_1^{-1}(e) \cap L = \emptyset$. This is a contradiction to part (1). \square

5 Symplectic del Pezzo surfaces

5.1 Lagrangian \mathbb{RP}^2 's in del Pezzo surfaces

Let (X_k, ω_0) be a symplectic del Pezzo surface, i.e. a monotone $\mathbb{CP}^2 \# k \overline{\mathbb{CP}}^2$ built by blowing up $k \leq 8$ disjoint balls $B(\frac{1}{3}) \subset (\mathbb{CP}^2, \omega)$ where ω is normalized so that $\int_{\mathbb{CP}^1} \omega = 1$.

Proposition 5.1. *There is a symplectic packing of 8 balls $B(\frac{1}{3})$ into $(\mathbb{CP}^2 \setminus \mathbb{RP}^2, \omega)$ and therefore there is a Lagrangian \mathbb{RP}^2 in any del Pezzo surfaces (X_k, ω_0) for $k \leq 8$.*

Proof. By Lemma 5.2 below it suffices to pack 8 balls of size $\frac{1}{3}$ into $(S^2 \times S^2, \Omega_{1, \frac{1}{2}})$. Since blowing up a ball of size $\frac{1}{3}$ in $(S^2 \times S^2, \Omega_{1, \frac{1}{2}})$ leads to $(\mathbb{CP}^2 \# 2 \overline{\mathbb{CP}}^2, \omega')$ with ω' dual to the class $\frac{7}{6}H - \frac{2}{3}E_1 - \frac{1}{6}E_2$, it suffices to prove that the vector

$$(7|4, 1, 2, 2, 2, 2, 2, 2) = 7H - 4E_1 - E_2 - 2 \sum_{i=3}^9 E_i$$

is Poincare dual to a symplectic form for $\mathbb{CP}^2 \# 9 \overline{\mathbb{CP}}^2$.

Permuting the E_j and applying the Cremona transform along $H - E_1 - E_2 - E_3$ three times, transforms $(7|4, 1, 2, 2, 2, 2, 2, 2)$ into $(4|2, 1, 1, 1, 1, 1, 1, 1)$, which is a reduced vector and it follows from the criterion in [26, Lemma 4.7(2)] that it represents a symplectic form. Similar arguments appeared in [4]. \square

Let $\Omega_{\alpha, \beta} = \alpha \sigma_A \oplus \beta \sigma_B$ be the symplectic form on $S^2 \times S^2$ where σ_A, σ_B are area forms on each factor with area 1. When $\alpha > 2\beta$, there is a symplectomorphism between $(S^2 \times S^2, \Omega_{\alpha, \beta})$ and the Hirzebruch surface $(H_4, \Omega_{\alpha, \beta})$. Here $H_4 = \mathbb{P}(\mathcal{O}(4) \oplus \mathbb{C}) \rightarrow \mathbb{CP}^1$, the fiber has area β , the zero section $Z_0 = \mathbb{P}(0 \oplus \mathbb{C})$ has area $\alpha + 2\beta$, and the section at infinity $Z_\infty = \mathbb{P}(\mathcal{O}(4) \oplus 0)$ has area $\alpha - 2\beta$, and the symplectomorphism identifies Z_∞ with a symplectic surface $Z \subset S^2 \times S^2$ in class $A - 2B$.

Lemma 5.2. *Symplectic ball packing for $(\mathbb{CP}^2 \setminus \mathbb{RP}^2, \omega)$ and $(S^2 \times S^2, \Omega_{1, \frac{1}{2}})$ are equivalent.*

Proof. It follows for the Biran decomposition [3, Theorem 1.A] associated to the quadric $Q \subset \mathbb{CP}^2$ [3, Section 3.1.2], that there is an open set in $(\mathbb{CP}^2 \setminus \mathbb{RP}^2, \omega)$ symplectomorphic to $(H_4 \setminus Z_\infty, \Omega_{\alpha, \beta})$ where $\alpha > 2\beta$ and $\alpha + 2\beta = 2$. Hence there is a symplectic embedding of $(S^2 \times S^2 \setminus Z, \Omega_\epsilon)$ into $(\mathbb{CP}^2 \setminus \mathbb{RP}^2, \omega)$, where $\Omega_\epsilon = \Omega_{\alpha, \beta}$ with $\alpha = 1 + 2\epsilon$ and $\beta = \frac{1}{2} - \epsilon$.

If $\phi : \coprod B_i \hookrightarrow (\mathbb{CP}^2 \setminus \mathbb{RP}^2, \omega)$ is a symplectic packing, then by the openness of symplectic ball-packing we also have an packing of $(1 + \delta) \coprod B_i$ for some $\delta > 0$. Performing a symplectic cut around \mathbb{RP}^2 gives a symplectic packing of $(1 + \delta) \coprod B_i$

into $(S^2 \times S^2, \Omega_\epsilon)$ and by rescaling we have a symplectic packing of $\coprod B_i$ into $(S^2 \times S^2, \frac{1}{1+\delta}\Omega_\epsilon)$. Taking $\epsilon = \delta/2$ gives a packing of $(S^2 \times S^2, \Omega_{1,c})$ where $c < 1/2$ and hence a packing of $\Omega_{1, \frac{1}{2}}$.

Conversely, let $\phi : \coprod B_i \hookrightarrow (S^2 \times S^2, \Omega_{1, \frac{1}{2}})$ be a symplectic packing and form a blow-up using ϕ . Standard Gromov–Witten theory shows that one may find a set of exceptional spheres in the complement of a curve in class B of the blown-up manifold. This implies the packing can be performed in the complement of a curve in class B and hence a packing of $(S^2 \times S^2, \Omega_{1, \frac{1}{2}-\epsilon})$ for some $\epsilon > 0$. This in turn gives a packing of $(S^2 \times S^2, \Omega_\epsilon)$. Using the relative inflation methods of [2] gives a packing of $(S^2 \times S^2, \Omega_\epsilon)$ in the complement of the (-4) -sphere Z and hence into an open subset of $(\mathbb{CP}^2 \setminus \mathbb{RP}^2, \omega)$. \square

5.2 The Proof of Theorem 1.5 and applications

The key to proving Theorem 1.5 is the following lemma.

Lemma 5.3. *Let $(\overline{M}^4, \overline{\omega})$ be a closed rational or ruled symplectic manifold with rational symplectic form $[\overline{\omega}] \in H^2(\overline{M}; \mathbb{Q})$. Suppose that $L \subset (\overline{M}, \overline{\omega})$ is a Lagrangian \mathbb{RP}^2 or S^2 , there are two orthogonal exceptional classes $E_1, E_2 \in \mathcal{E}_{\overline{\omega}}$ with the same symplectic area a , and $\langle [E_i], [L] \rangle_{\mathbb{Z}} = 0$. Then there is a Lagrangian sphere $L_0 \subset (\overline{M}, \overline{\omega})$ in class $E_1 - E_2$ that is disjoint from L .*

Proof. Notice that the homological pairing condition is void when $L = \mathbb{RP}^2$. We only give the proof for \mathbb{RP}^2 , and the case for $L = S^2$ is identical with the trivial pairing assumption.

By Proposition 4.1 we can pick representatives C_i for the classes E_i that are disjoint from L . Hence performing a symplectic cut on a neighborhood of L produces $(\overline{W}, \overline{\omega})$ with cut divisor Z disjoint from the exceptional spheres C_i . Let (W, ω) be the result of blowing down the two curves C_i to balls $B_i(a)$, so that $B_i(a)$ are disjoint from $Z \subset W$, and let $p : \overline{W} \rightarrow W$ be the topological blow down map.

Taking sub-balls $B_i(\epsilon) \subset B_i(a)$ for ϵ sufficiently small, we can assume there is a Darboux chart that is disjoint from Z and contains both $B_i(\epsilon)$. If $(\overline{W}, \overline{\omega}_\epsilon)$ is the result of blowing-up these two balls, it follows from the local construction in the proof of [28, Lemma 5.4] that there is a Lagrangian sphere $L'_0 \subset (\overline{W}, \overline{\omega}_\epsilon)$ that is disjoint from Z and in class $E_1 - E_2$.

Let $B_b = PD(p^*[\omega]) - b(E_1 + E_2)$ where b is a rational number slightly larger than a , then by [28, Lemma 5.1] and [2, Lemma 2.2.B] for $q \gg 0$ sufficiently large we have a symplectic surface $C \subset (\overline{W}, \overline{\omega}_\epsilon)$ such that $[C] = qB_b$, is disjoint from the Lagrangian sphere L'_0 , and intersects Z transversally. Biran's relative inflation method [2, Proposition 2.1.A] along C now builds a symplectic form $\overline{\omega}' \in [\overline{\omega}]$ on \overline{W} such that L'_0 is $\overline{\omega}'$ -Lagrangian and Z is $\overline{\omega}'$ -symplectic. Since by construction $\overline{\omega}'$ and $\overline{\omega}$ are deformation equivalent, through symplectic forms for which Z is a symplectic manifold it follows from [36, Proposition 1.2.9] that there is an symplectomorphism

$\phi : (\overline{W}, \bar{\omega}') \rightarrow (\overline{W}, \bar{\omega})$ that preserves Z . Hence $L_0 = \phi(L'_0)$ is a Lagrangian sphere in $(\overline{W}, \bar{\omega})$ that is disjoint from Z and is in homology class $E_1 - E_2$. Gluing a neighborhood of \mathbb{RP}^2 back in for Z gives the desired result. \square

Proof of Theorem 1.5. Since we are in the monotone case, all Lagrangian spheres $S^2 \subset X_k$ are heavy with respect to the fundamental class $\mathbb{1} \in QH_4(X_k; \mathbb{Z}/2\mathbb{Z})$ by [10, Theorem 1.17]. So to show that a given Lagrangian L is not superheavy with respect to the fundamental class $\mathbb{1}$, by [10, Theorem 1.4] it suffices to build a Lagrangian sphere L' disjoint from L .

Let L be a Lagrangian \mathbb{RP}^2 in (X_k, ω_0) with $k \geq 2$, then by Lemma 5.3 there is a non-trivial Lagrangian sphere $L_0 \subset X_k$ that is disjoint from L .

Let L be a Lagrangian S^2 in (X_k, ω_0) with $k \geq 3$. By [28, Theorem 1.4] and Lemma 5.3 there are Lagrangian spheres L_1 and L_2 that are disjoint from each other. Using [28, Theorem 1.8] and Corollary 1.2, we have a symplectomorphism ψ such that $\psi(L_2) = L$ and therefore $\psi(L_1)$ is a Lagrangian S^2 disjoint from L . \square

Remark 5.4. *The case when $k = 2$ for Lagrangian spheres can be proved by adapting the computation of Fukaya–Oh–Ohta–Ono [13] in the $S^2 \times S^2$ case to $S^2 \times S^2 \# \overline{\mathbb{CP}}^2 = (X_2, \omega_2)$. Therefore combining the result for $k = 0$ proved in [43], monotone spherical Lagrangians in rational surfaces are proved to not be superheavy, with the exception of $\mathbb{RP}^2 \subset \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$, which presumably can be done using techniques in [43].*

We discuss an application of Theorem 1.5 of potential interest. We will consider the toric degeneration picture Figure 1 of $S^2 \times S^2$ that first appeared in [13], but let us note that it is possible for one to consider other semi-toric systems as well (cf. [43]).

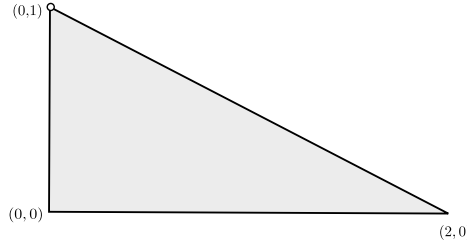


Figure 1: $S^2 \times S^2$

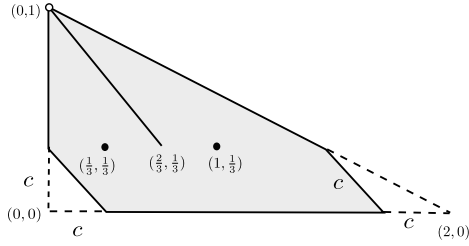


Figure 2: $S^2 \times S^2 \# 2\overline{\mathbb{CP}}^2$

The reader should consider Figure 1 as an “open polytope” in the sense that there is no actual toric action on $S^2 \times S^2$ with the polytope in Figure 1. Instead, there is such a toric action on the complement of the anti-diagonal $S^2 \times S^2 \setminus \Delta$, whose moment map has image equal the polytope in Figure 1 with the white dot removed.

In this open toric picture, which is closely related to the *semi-toric* case considered by [39], one could still perform two toric blow-ups of size c by standard “corner chopping”, giving Figure 2 as moment polytope. When $c = \frac{1}{3}$, the resulting symplectic manifold $S^2 \times S^2 \# 2\overline{\mathbb{CP}}^2$ is monotone and symplectomorphic to X_3 in Theorem 1.5. Since the proper transform of the Lagrangian $\bar{\Delta} \subset S^2 \times S^2$ is not superheavy it follows that some fiber in the open polytope must be non-displaceable by [10, Theorem 1.8]. One may further narrow down the position of these fibers using probes invented by McDuff [35]. It is not hard to check that there are three sets of potentially non-displaceable fibers as shown by the solid line and two extra dots in Figure 2. What we showed is that at least one of these points is actually non-displaceable.

A Lagrangian spheres in rational manifolds

A.1 Characteristic Lagrangian spheres

We first recall from [28, Definition 3.3] that a *stable spherical symplectic configuration* is an ordered configuration of symplectic spheres with: (1) $c_1 \geq 1$ for all irreducible components, (2) the intersection numbers between two different components are 0 or 1, (3) they are simultaneously holomorphic with respect to some almost complex structure J tamed by the symplectic form. We will call them stable configurations for brevity. In the proof of [28, Theorem 1.5], the following intermediate result is reached.

Lemma A.1. *In $\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2$, let L_1 and L_2 be Lagrangian spheres in the homology class $E_1 - E_2$ and suppose they are disjoint from a stable configuration with irreducible components in classes $\{H - E_1 - E_2, H - E_3 - E_4, E_3, E_4\}$, then L_1 and L_2 are Hamiltonian isotopic in the complement of the stable configuration.*

In particular in the proof of [28, Theorem 1.5] one uses [28, Proposition 6.8] to show that L_1 and L_2 are Hamiltonian isotopic in the complement of the stable configuration. The same holds true for $\mathbb{CP}^2 \# (k+1)\overline{\mathbb{CP}}^2$ as well for $k = 1, 2$ with the stable configurations specified in [28].

Theorem A.2. *Lagrangian S^2 ’s in a symplectic rational manifold with $\chi \leq 7$ are unique up to Hamiltonian isotopy.*

Proof. By [28, Theorem 1.5 and Proposition 4.10], we only need to deal with the case where $M = \mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ and $[L_i] = H - E_1 - E_2 - E_3$ for $i = 1, 2$.

Fix a Darboux chart $U_p \subset M$ that is disjoint from $L_1 \cup L_2$ and centered at the point $p \in M$. By blowing-up a symplectically embedded ball $B_p \subset U_p$, we can build a symplectic manifold $(M' = \mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2, \omega')$ with a exceptional sphere C such that $H_2(M'; \mathbb{Z})$ has a basis identified with the union of a basis of $H_2(M, \mathbb{Z})$ and $[C]$, the intersection product $[L_i] \cap [C] = 0$.

From the Gromov–Taubes invariant theory, for generic compatible almost complex structure J the classes $H - E_1 - [C]$, $H - E_2 - [C]$, and $H - E_3 - [C]$ have unique representatives as J -holomorphic exceptional spheres C_1 , C_2 and C_3 , respectively, which are disjoint. Since $[C_i] \cap [L_j] = 0$, [28, Corollary 3.13] builds Hamiltonian isotopies ψ_j so that $\psi_j(L_j)$ is disjoint from $C_1 \cup C_2 \cup C_3 \cup C$.

Notice that the set of classes $\{H - E_1 - [C], H - E_2 - [C], H - E_3 - [C], [C]\}$ are Cremona equivalent to $\{H - E_1 - E_2, H - E_3 - E_4, E_3, E_4\}$, Lemma A.1 applies. It follows that L_1 and L_2 are Lagrangian isotopic in the complement of a neighborhood of $C \cup \bigcup C_i$ in (M', ω') , in particular the complement of C which is symplectomorphic to an open set of M . \square

A.2 Proof of Corollary 1.2

Proof. Part (2) follows from Theorem A.2 and [28, Theorem 1.6]. When $\chi(M) = 6$ and the homology class of the Lagrangians is characteristic, then Theorem A.2 covers part (1). In all the rest of cases, we assume that $[L_i] = E_1 - E_2$ without loss of generality by [28, Proposition 4.10]. Our proof follows the steps sketched in [28].

For each pair (M, L_i) by [28, Theorem 1.1], away from L_i there is a set of disjoint (-1) symplectic spheres C_i^l for $l = 3, \dots, k+1$, with

$$[C_i^l] = E_l \text{ for } l = 3, \dots, k \text{ and } [C_i^{k+1}] = H - E_1 - E_2.$$

Blowing down the collections $\mathcal{C}_i = (C_i^3, \dots, C_i^{k+1})$ separately, results in $(\tilde{M}_i, \tilde{L}_i, \mathcal{B}_i)$ where \tilde{M}_i is a symplectic $S^2 \times S^2$ with equal symplectic areas in each factor, \tilde{L}_i a Lagrangian sphere, and $\mathcal{B}_i = (B_i^3, \dots, B_i^{k+1})$ is a symplectic ball packing in $\tilde{M}_i \setminus \tilde{L}_i$ corresponding to \mathcal{C}_i .

By Lalonde–McDuff [22] and Hind [16], there is a symplectomorphism between the pairs $\Psi : (\tilde{M}_1, \tilde{L}_1) \rightarrow (\tilde{M}_2, \tilde{L}_2)$. For fixed l , the symplectic balls $\Psi(B_1^l)$ and B_2^l have the same volume since they come from blowing down the same class. Hence by Theorem 1.1 there is a compactly supported Hamiltonian isotopy Φ of $\tilde{M}_2 \setminus \tilde{L}_2$ connecting the symplectic ball packing $\Psi(\mathcal{B}_1) = \{\Psi(B_1^l)\}_l$ and \mathcal{B}_2 in $\tilde{M}_2 \setminus \tilde{L}_2$. Therefore $\Phi \circ \Psi$ is a symplectomorphism between the tuples $(\tilde{M}_i, \tilde{L}_i, \mathcal{B}_i)$ and hence upon blowing up induces a symplectomorphism

$$\psi : (M, L_1, \mathcal{C}_1) \rightarrow (M, L_2, \mathcal{C}_2).$$

By design ψ preserves the homology classes $E_1 - E_2$, $H - E_1 - E_2$, E_3, \dots, E_k as well as the class $[\omega]$, from which it follows that ψ it also preserves H and hence $\psi \in \text{Symp}_h(M)$. \square

B \mathbb{Z}_2 -orthogonal systems and push off systems

Recall that in the proof of Lemma 4.8 for $1 \leq k \leq 8$ we need to compare the counts for:

- \mathbb{Z}_2 -orthogonal systems for H , i.e. sets of k exceptional classes that are \mathbb{Z}_2 -orthogonal to H in $\mathbb{CP}^2 \# k\overline{\mathbb{CP}}^2$.
- Push off system for S , i.e. sets of k exceptional classes that are \mathbb{Z} -orthogonal to $S = -H + 2E_1 - E_2$ in $\mathbb{CP}^2 \# (k+1)\overline{\mathbb{CP}}^2$.

For ease of notation when $k \geq 4$ we will replace S with the Cremona equivalent $S' = E_1 - E_2 - E_3 - E_4$. In the following $\mathcal{O}_k = \{E_i\}_{i=1}^k$ will be a \mathbb{Z}_2 -orthogonal system for H in $\mathbb{CP}^2 \# k\overline{\mathbb{CP}}^2$, while $\mathcal{P}_k = \{H - E_1 - E_i\}_{i=2}^4 \cup \{E_j\}_{j=5}^{k+1}$ will be a push off system for S' in $\mathbb{CP}^2 \# (k+1)\overline{\mathbb{CP}}^2$.

For $k \leq 3$: H has \mathcal{O}_k and S has $\{H - E_1 - E_2\} \cup \{E_i\}_{i=3}^{k+1}$.

For $4 \leq k \leq 5$: H has \mathcal{O}_k and S' has \mathcal{P}_k .

For $k = 6$: H has \mathcal{O}_6 and its Cremona transform along $2H - \sum_{i=1}^6 E_i$. While S' has \mathcal{P}_6 and its Cremona transform along $H - E_5 - E_6 - E_7$.

For $k = 7$: There are 8 systems: H has \mathcal{O}_7 and its 7 Cremona transforms along

$$2H - (E_1 + \cdots + E_7) + E_i \quad \text{for } 1 \leq i \leq 7.$$

While S' has \mathcal{P}_7 , its 4 Cremona transforms along

$$H - (E_5 + E_6 + E_7 + E_8) + E_j \quad \text{for } j = 5, 6, 7, 8$$

and its 3 Cremona transforms along

$$2H - E_1 - (E_5 + E_6 + E_7 + E_8) + E_j \quad \text{for } j = 2, 3, 4.$$

For $k = 8$: There are 29 systems: H has \mathcal{O}_8 and its 28 Cremona transforms along

$$2H - (E_1 + \cdots + E_7 + E_8) + E_i + E_j \quad \text{for } 1 \leq i \neq j \leq 8.$$

While S' has \mathcal{P}_8 , its 10 Cremona transforms along

$$H - (E_5 + E_6 + E_7 + E_8 + E_9) + E_i + E_j \quad \text{for } 5 \leq i \neq j \leq 9,$$

its 15 Cremona transforms along

$$2H - E_1 - (E_5 + E_6 + E_7 + E_8 + E_9) + E_j + E_p \quad \text{for } 2 \leq j \leq 4, 5 \leq p \leq 9,$$

and its 3 Cremona transforms along

$$3H - 2E_1 - (E_2 + E_3 + E_4) - (E_5 + E_6 + E_7 + E_8 + E_9) + E_q \quad \text{for } 2 \leq q \leq 4.$$

References

- [1] M. Audin. Lagrangian skeletons, periodic geodesic flows and symplectic cuttings. *Manuscripta Math.*, 124(4):533–550, 2007.
- [2] P. Biran. A stability property of symplectic packing. *Invent. Math.*, 136(1):123–155, 1999.
- [3] P. Biran. Lagrangian barriers and symplectic embeddings. *Geom. Funct. Anal.*, 11(3):407–464, 2001.
- [4] O. Buse and M. Pinsonnault. Packing numbers of rational ruled 4-manifolds. *J. Symplectic Geom. (to appear)*, 2013. arXiv:1104.3362v1.
- [5] G. Dimitroglou Rizell and J.D. Evans. Unlinking and unknottedness of monotone Lagrangian submanifolds. arXiv:1211.6633v1, 2012.
- [6] J.G. Dorfmeister. Kodaira dimension of fiber sums along spheres. arXiv:1008.4447v1, 2010.
- [7] J.G. Dorfmeister. Minimality of symplectic fiber sums along spheres. *Asian J. Math. (to appear)*, 2013. arXiv:1005.0981v1.
- [8] R. Fintushel and R.J. Stern. Invariants for Lagrangian tori. *Geom. Topol.*, 8:947–968 (electronic), 2004.
- [9] Y. Eliashberg and L. Polterovich. The problem of Lagrangian knots in four-manifolds. In *Geometric topology (Athens, GA, 1993)*, volume 2 of *AMS/IP Stud. Adv. Math.*, pages 313–327. Amer. Math. Soc., Providence, RI, 1997.
- [10] M. Entov and L. Polterovich. Rigid subsets of symplectic manifolds. *Compos. Math.*, 145(3):773–826, 2009.
- [11] J.D. Evans. Lagrangian spheres in del Pezzo surfaces. *J. Topol.*, 3(1):181–227, 2010.
- [12] J.D. Evans. Symplectic mapping class groups of some Stein and rational surfaces. *J. Symplectic Geom.*, 9(1):45–82, 2011.
- [13] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono. Toric Degeneration and nondisplaceable Lagrangian tori in $S^2 \times S^2$. *Int. Math. Res. Not. IMRN*, 2012(13):2942–2993, 2012.
- [14] R.E. Gompf. A new construction of symplectic manifolds. *Ann. of Math. (2)*, 142(3):527–595, 1995.
- [15] J.-C. Hausmann and A. Knutson. Cohomology rings of symplectic cuts. *Differential Geom. Appl.*, 11(2):197–203, 1999.

- [16] R. Hind. Lagrangian spheres in $S^2 \times S^2$. *Geom. Funct. Anal.*, 14(2):303–318, 2004.
- [17] R. Hind. Lagrangian spheres in Stein surfaces. *Asian J. Math.*, 16(1):1–36, 2012.
- [18] E.-N. Ionel and T.H. Parker. The symplectic sum formula for Gromov-Witten invariants. *Ann. of Math. (2)* 159(3):935–1025, 2004.
- [19] M. Kervaire and J.W. Milnor. On 2-spheres in 4-manifolds. *Proc. Nat. Acad. Sci. USA*, 47:1651–1657, 1961.
- [20] M. Khovanov and P. Seidel. Quivers, Floer cohomology, and braid group actions. *J. Amer. Math. Soc.*, 15(1):203–271, 2002.
- [21] F. Lalonde. Isotopy of symplectic balls, Gromov’s radius and the structure of ruled symplectic 4-manifolds. *Math. Ann.*, 300(2):273–296, 1994.
- [22] F. Lalonde and D. McDuff. J -curves and the classification of rational and ruled symplectic 4-manifolds. In *Contact and symplectic geometry (Cambridge, 1994)*, volume 8 of *Publ. Newton Inst.*, pages 3–42. Cambridge Univ. Press, Cambridge, 1996.
- [23] E. Lerman. Symplectic cuts. *Math. Res. Lett.*, 2(3):247–258, 1995.
- [24] B.-H. Li and T.-J. Li. Symplectic genus, minimal genus and diffeomorphisms. *Asian J. Math.*, 6(1):123–144, 2002.
- [25] B.-H. Li and T.-J. Li. Smooth minimal genera for small negative classes in $\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}$ with $n \leq 9$. *Topology Appl.*, 6(1):123–144, 2002.
- [26] T.-J. Li and A.-K. Liu. Uniqueness of symplectic canonical class, surface cone and symplectic cone of 4-manifolds with $B^+ = 1$. *J. Differential Geom.*, 58(2):331–370, 2001.
- [27] A.-M. Li and Y. Ruan. Symplectic surgery and Gromov-Witten invariants of Calabi-Yau 3-folds. *Invent. Math.*, 145(1):151–218, 2001.
- [28] T.-J. Li and W. Wu. Lagrangian spheres, symplectic surfaces and the symplectic mapping class group. *Geom. Topol.*, 16(2):1121–1169, 2012.
- [29] A. Liu. Some new applications of the general wall crossing formula. *Math. Res. Lett.*, 3:569–585, 1996.
- [30] D. McDuff. The structure of rational and ruled symplectic 4-manifolds. *J. Amer. Math. Soc.*, 3(3):679–712, 1990.
- [31] D. McDuff. Blow ups and symplectic embeddings in dimension 4. *Topology*, 30(3):409–421, 1991.

- [32] D. McDuff. Remarks on the uniqueness of symplectic blowing up. In *Symplectic geometry*, volume 192 of *London Math. Soc. Lecture Note Ser.*, pages 157–167. Cambridge Univ. Press, Cambridge, 1993.
- [33] D. McDuff. Lectures on Gromov invariants for symplectic 4-manifolds. In *Gauge theory and symplectic geometry (Montreal, PQ, 1995)*, volume 488 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 175–210. Kluwer Acad. Publ., Dordrecht, 1997.
- [34] D. McDuff. From symplectic deformation to isotopy. In *Topics in symplectic 4-manifolds (Irvine, CA, 1996)*, First Int. Press Lect. Ser., I, pages 85–99. Int. Press, Cambridge, MA, 1998.
- [35] D. McDuff. Displacing Lagrangian toric fibers via probes. In *Low-dimensional and symplectic topology*, volume 82 of *Proc. Sympos. Pure Math.*, pages 131–160. Amer. Math. Soc., Providence, RI, 2011.
- [36] D. McDuff. Nongeneric J -holomorphic curves in rational manifolds. arXiv:1211.2431v1, 2012.
- [37] D. McDuff and L. Polterovich. Symplectic packings and algebraic geometry. *Invent. Math.*, 115(3):405–434, 1994. With appendix by Y. Karshon.
- [38] D. McDuff and F. Schlenk. The embedding capacity of 4-dimensional symplectic ellipsoids. *Ann. of Math. (2)*, 175(3):1191–1282, 2012.
- [39] A. Pelayo and S. Vũ Ngoc. Semitoric integrable systems on symplectic 4-manifolds. *Invent. Math.*, 177(3):571–597, 2009.
- [40] P. Seidel. Lectures on four-dimensional Dehn twists. In *Symplectic 4-manifolds and algebraic surfaces*, volume 1938 of *Lecture Notes in Math.*, pages 231–267. Springer, Berlin, 2008.
- [41] S. Vidussi. Lagrangian surfaces in a fixed homology class: existence of knotted Lagrangian tori. *J. Differential Geom.*, 74(3):507–522, 2006.
- [42] C. H. Taubes. $SW \Rightarrow Gr$: from the Seiberg-Witten equations to pseudo-holomorphic curves. *J. Amer. Math. Soc.*, 9(3):845–918, 1996.
- [43] W. Wu. On an exotic Lagrangian torus in $\mathbb{C}P^2$. arXiv:1201.2446, 2012.
- [44] W. Wu. Uniqueness of Lagrangian spheres in A_n -singularities. Work in progress.

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